

A new representation of Links: Butterflies

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ABSTRACT. With the idea of an eventual classification of 3-bridge links, we define a very nice class of 3-balls (called butterflies) with faces identified by pairs, such that the identification space is S^3 , and the image of a preferred set of edges is a link. Several examples are given. We prove that every link can be represented in this way (butterfly representation). We define the butterfly number of a link, and we show that the butterfly number and the bridge number of a link coincide. This is done by defining a move on the butterfly diagram. We give an example of two different butterflies with minimal butterfly number representing the knot 8_{20} . This raises the problem of finding a set of moves on a butterfly diagram connecting diagrams representing the same link. This is left as an open problem.

1. Introduction

The beautiful classification of 2-bridge links by rational numbers has not yet been generalized to 3-bridge links. One of the goals of this paper is to introduce a tool that eventually might lead to a generalization of this classification.

It is well known [19] that every closed, orientable 3-manifold can be obtained by pasting pairs of faces of a polygonization of the boundary S^2 of a closed 3-cell \mathbf{B}^3 .

Thurston's construction of the borromean rings, [24] and [25], is a nice example that we generalize for all links in this paper, Fig. 1. In this example we notice that the cube is actually a closed 3-cell \mathbf{B}^3 , with twelve faces on its boundary that are identified by reflections along some axes (double arrows). Moreover, pasting the faces of the cube we obtain S^3 and the set of axes become the borromean rings.

These reflections resemble the way a butterfly closes its wings, and we will say that the borromean rings have a 6-butterfly representation, and the six faces of the real cube are the six butterflies involved.

Similarly to the borromean rings, the 2-bridge knots or rational links p/q can be obtained by pasting the northern and southern hemispheres of S^2 with themselves by reflections through half meridians separated apart $2\pi q/p$. For instance, Fig. 22 depicts this construction for $p/q = 3/1$, the trefoil knot. As in Thurston's example,

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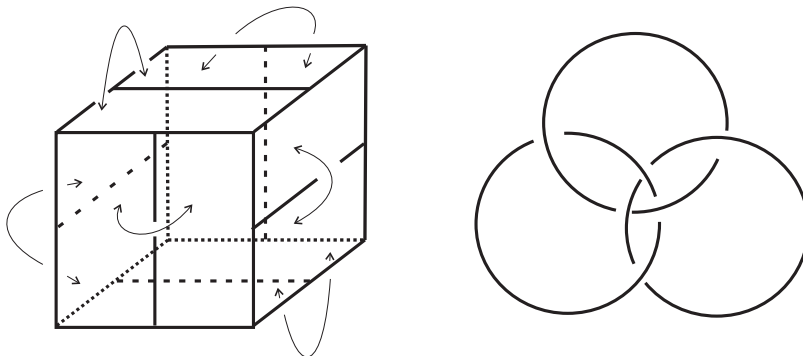


FIGURE 1. Borromean rings.

S^3 is obtained by pasting the faces. We say that the rational link p/q has a 2-butterfly representation, and the northern and southern hemispheres of S^2 are the two butterflies involved.

This butterfly representation of p/q has two main advantages. First, it is a pure 2-dimensional diagram, and secondly, it exhibits directly the rational number p/q that classifies the knot or link.

With these two properties in mind, we wondered if all knots and links have a similar structure, allowing two or more butterflies on the boundary S^2 of \mathbf{B}^3 . One such structure with three butterflies is depicted in Fig. 5b.

It turns out that every knot or link admits such a representation. We prove this fact here. In Sections 4 and 5 we give algorithms to pass back and forth from a link to a butterfly representation of it.

We define accordingly the butterfly number of a knot or link and we prove that it coincides with its bridge number (Section 7). To obtain this last result we need to reduce the number of butterflies of a particular butterfly representation of a link. This involves the definition of a move that does precisely this. See Section 6.

As each m -bridge link diagram has an m -butterfly representation, a natural question arises: *Is it possible to associate a set of rational numbers to describe this butterfly?* In the case $m = 3$ this assignation can, in fact, be made [11], where a triple of rational numbers is associated to each 3-butterfly. In this paper we show some examples of 3-butterflies and its corresponding set of rational numbers.

We give many examples and in particular two different 3-butterfly representations of the same knot 8_{20} . This raises the problem of relating 3-butterfly representations by a set of potential moves. This is left as an open problem. Using the concept of 3-butterfly, we hope to obtain a classification of 3-bridge links, similar to the Schubert classification of 2-bridge links.

In Section 2 we present a technical definition of an m -butterfly even though in the rest of the paper, for simplicity, we speak more intuitively about m -butterflies.

In the last decade, Kauffman [13] has been developing the theory of virtual knots. This theory has several applications. The technical definition of an m -butterfly is used intensively in [12] where we prove that any virtual knot also admits a representation by a generalized (n, g) -butterfly, that is a handlebody of genus g with $2n$ faces on its boundary that are identified by reflections along some axis.

As we have remarked above, pasting the faces of an m -butterfly gives the 3-sphere S^3 . Section 3 is devoted to showing this fact. In general, this result is not true for generalized (n, g) -butterflies that represent virtual knots.

2. Butterflies: Definitions and Examples

Intuitively, an m -butterfly is a 3-ball \mathbf{B}^3 with $m > 0$ polygonal faces on its boundary $S^2 = \partial\mathbf{B}^3$, such that each face P is subdivided by an arc t_P in two subfaces (that have the same number of vertices) that are identified by a "reflection" along this arc t_P .

In order to formalize this concept, we give some technical definitions.

Let R be a connected graph embedded in $S^2 = \partial\mathbf{B}^3$, where \mathbf{B}^3 is a closed 3-cell, so that $S^2 - R$ is a disjoint union of open 2-cells. For our purposes we assume that \mathbf{B}^3 is the half ball $x^2 + y^2 + z^2 \leq r^2; z \leq 0$, and that the graph R and later the graph $R \cup T$, when T has been defined, is contained in the planar part of $\mathbf{B}^3, \mathbb{R}^2 \times \{0\}$. The edges in R and in T are simple arcs. However, by [4], for any such graph $R \cup T$ there is an autohomeomorphism of S^2 such that the images of the edges are straight planar line segments. We shall assume, in the proofs of theorems that follow, but not in the drawn figures, that the edges of $R \cup T$ are straight planar line segments. We denote each open 2-cell generically by P . We would like to parameterize each 2-cell P .

For any $n \in \mathbb{N}$, let P_{2n} be the regular polygon that is the closed convex hull of the $2n^{\text{th}}$ roots of unity. We define a parameterization of P to be a function f from P_{2n} to the closure \overline{P} of P , with the following properties:

- a) The restriction of f to interior P_{2n} is a homeomorphism from interior P_{2n} to P .
- b) The restriction of f to an edge of P_{2n} is a piecewise linear homeomorphism from that edge to an edge in the graph R .
- c) f as a map from the edges of ∂P_{2n} to the edges of ∂P is at most 2 to 1.

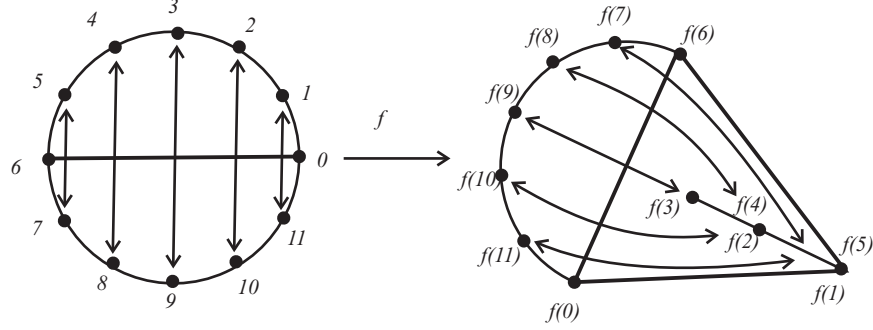
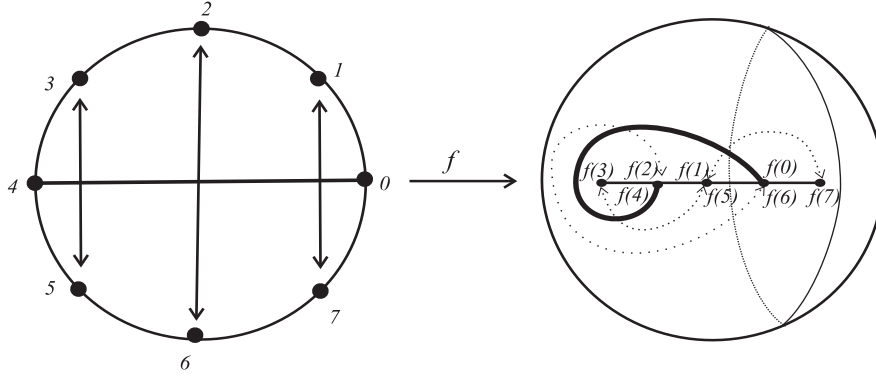
The existence of a parameterization of P places restrictions on P and on R . We will assume that R is such that each P has a parameterization $f : P_{2n} \rightarrow \overline{P}$, and we fix a parameterization f_P for each P .

Complex conjugation, $z \rightarrow \overline{z}$, restricted to P_{2n} or to boundary of P_{2n} defines an involution and an equivalence relation on the edges and vertices of P_{2n} , and this in turn, induces an equivalence relation on the edges and vertices of \overline{P} , and on the points of P as well. That is to say for A and B points of \overline{P} , $A \sim B$ if $f_P^{-1}(A) = f_P^{-1}(B)$ or $f_P^{-1}(A) = \overline{f_P^{-1}(B)}$, where $\overline{f_P^{-1}(B)} = \{\overline{z}/z \in f_P^{-1}(B)\}$.

The equivalence relation on the edges and vertices of each \overline{P} induces an equivalence relation on the graph R . That is $x \simeq y$ if and only if there exists a finite sequence $x = x_1, \dots, x_l = y$ with $x_i \sim x_{i+1}$ for $i = 1, \dots, l-1$. Equivalence classes of points of P contain two points except for those points in $f([-1, 1])$ where there is only one point. Note that if x is a vertex of R , its complete class under the equivalence relation \simeq is composed entirely of vertices.

Figures 2 and 3 illustrate two different parameterizations. In Fig. 2 we have $f(1) = f(5)$ and $f(2) = f(4)$; and in Fig. 3 we have $f(0) = f(6)$, $f(1) = f(5)$ and $f(2) = f(4)$.

Each P_{2n} contains the line segment $[-1, 1]$ which is the fixed point set of complex conjugation restricted to P_{2n} . The image of this line segment $f_P([-1, 1])$ is called the *trunk* t . A pair (P, t) will be called a *butterfly with trunk* t . The *wings* W

FIGURE 2. f parameterizes a pair (P, t) .FIGURE 3. f parameterizes a 1-butterfly.

and W' are just $f_P(P_{2n} \cap \text{upper half plane})$ and $f_P(P_{2n} \cap \text{lower half plane})$ and $W \cap W' = t$. Each time that we consider a trunk t we are implicitly considering the equivalence relation described above. We denote by T the collection of all trunks t (over all P). Notice that the boundaries of the n butterflies form a graph R on $S^2 = \partial \mathbf{B}^3$. As before, (See [4]), we can assume the edges in the graph $R \cup T$ as straight line segments.

Let us denote by $M(R, T)$ the space \mathbf{B}^3 / \simeq with the topology of the identification map $p : \mathbf{B}^3 \rightarrow M(R, T)$.

As in Thurston's example, we would like that the image of $T, p(T)$, became a knot or link. In order to guarantee this fact, we distinguish three types of vertices on R .

A member of $R \cap T$ will be called an *A-vertex*. A member of $p^{-1}(p(v))$, $v \in R \cap T$, which is not an *A-vertex* will be called an *E-vertex*. A vertex of R which is not an *A-vertex* nor an *E-vertex* will be called a *B-vertex* iff $p^{-1}(p(v))$ contains at least one non-bivalent vertex of R .

We do not give an explicit name for those vertices that are neither *A, B* nor *E-vertices*. Of course it is possible to construct 3-balls with polygonal faces on their boundaries with those kind of vertices but for our purposes (we want to represent knots or links) it is enough to consider graphs without them. There are

also interesting examples in which there are E -vertices that are not bivalent, as the one shown in Fig. 4, but for our purpose we do not consider them as m -butterflies. In further research we will consider some generalization of our construction.

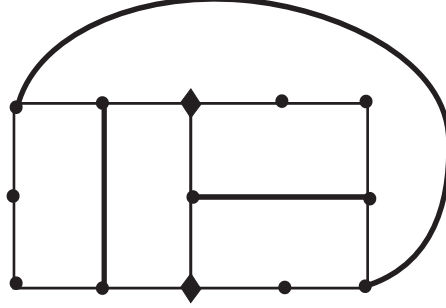


FIGURE 4. In this polygonization, the vertices marked with \blacklozenge are trivalent E -vertices of R .

With these definitions we formalize our intuitive definition of m -butterfly, given at the beginning of this section.

DEFINITION 1. For $m \geq 1$, an m -butterfly is a 3-ball \mathbf{B}^3 with m butterflies (P, t) on its boundary $S^2 = \partial\mathbf{B}^3$, such that (i) the graph R has only A -vertices, E -vertices and B -vertices; (ii) the A - and E -vertices are bivalent in R , and (iii) T has m components.

Moreover, an m -butterfly can be represented by a planar graph (or by an m -butterfly diagram), denoted by a pair (R, T) , such that conditions (i), (ii), and (iii) are satisfied. The m -butterfly represented by the diagram (R, T) is also denoted by (R, T) .

EXAMPLE 1. Figure 5 depicts three different butterfly diagrams. Fig 5b represents a 3-butterfly. The full equivalence class of the two trivalent vertices 0 and ∞ on it are B -vertices. Fig. 5a shows a 2-butterfly that has only A or E -vertices, while the 1-butterfly given in 5c has only two A -vertices and three B -vertices.

In the examples of Fig. 5 we will assume that \mathbf{B}^3 is the closed 3-cell that lies over the paper in $\mathbb{R}^3 + \infty$. The members of T will be displayed as thick lines. The B -vertices are depicted by $*$. See 5b and c. The other vertices of the diagram are either boundaries of members of T (A -vertices) or E -vertices.

3. The Quotient Space $M(R, T)$ is S^3

In this section we are going to prove that under our definitions, the space $M(R, T)$ is S^3 and that the image of T under the identification map p is a knot (or link). So we are sure to obtain a knot (or link) inside S^3 when we make the identifications by the equivalence relation.

THEOREM 1. For any m -butterfly (R, T) , the space $M(R, T)$ is homeomorphic to S^3 and $p(T)$ is a knot or a link, where $p : \mathbf{B}^3 \rightarrow M(R, T)$ is the identification map.

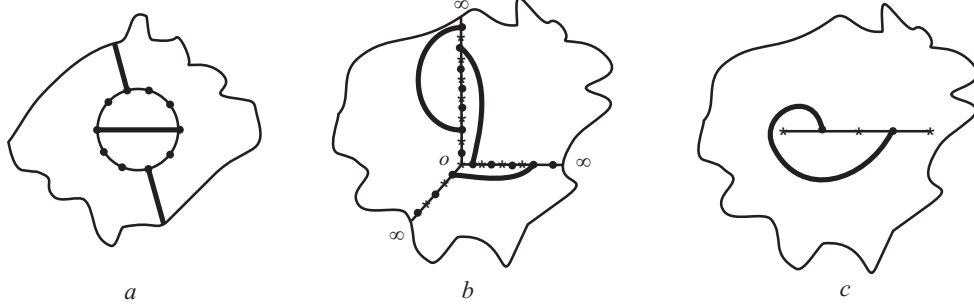


FIGURE 5. Representing butterflies with planar graphs.

PROOF. Set $M = M(R, T)$ for shortness. Let $R^* = p(R)$, $T^* = p(T)$ and $V^* = p(V)$, where V is the set of vertices of R . Let $U(V^*)$ be a regular neighbourhood of V^* in the space $M = M(R, T)$. Then $U(V^*)$ is a disjoint union of regular neighbourhoods (we choose $U(V^*)$ as small as we need) of the vertices of V^* . Let $v^* \in V^*$ be one of these vertices. Of course any regular neighbourhood of v^* is the cone over an orientable surface Σ_{v^*} .

Claim 1: The surface Σ_{v^*} is connected.

Proof: Consider the subset $p^{-1}(v^*)$ of the set V . Let $v \in p^{-1}(v^*)$. A regular neighbourhood of v in \mathbf{B}^3 is a cone from v over a 2-disk Δ_v properly embedded in \mathbf{B}^3 . Denote this cone by $C(v, \Delta_v)$. It is possible to select the regular neighbourhood of members of $p^{-1}(v^*)$ so that

$$\Sigma_{v^*} = \bigcup_{v \in p^{-1}(v^*)} p(\Delta_v).$$

Now, if $v_1, v_2 \in p^{-1}(v^*)$ then $v_1 \simeq v_2$, so there exists a finite sequence of vertices of $p^{-1}(v^*)$ say $u_1 = v_1, u_2, \dots, u_k = v_2$ such that $u_i \sim u_{i+1}, i = 1, \dots, k-1$. If we assume that u_i, u_{i+1} belong to some \overline{P} , where (P, t) is the corresponding butterfly, then the boundary of $\Delta_{u_i} \cap P$ and $\Delta_{u_{i+1}} \cap P$ are also identified and it follows that $p(\Delta_{u_i}) \cup p(\Delta_{u_{i+1}})$ is a connected set. From this, the claim follows easily.

We continue with the proof of the theorem. The closure of $M \setminus U(V^*)$ is clearly a compact, connected 3-manifold M^* with boundary $\partial M^* = \bigcup_{v^* \in V^*} \Sigma_{v^*}$. The closure in M^* of the set $R^* \setminus U(V^*)$ (resp. $T^* \setminus U(V^*)$) is a set of disjoint, properly embedded arcs in M^* that will be denoted by R^{**} (resp. T^{**}).

Now drill from M^* a regular neighbourhood $U(R^{**}) \cup U(T^{**})$ of $R^{**} \cup T^{**}$ and take the closure M^{**} of the result. Then M^{**} is the image under p of the ball $C = \mathbf{B}^3 \setminus U(R \cup T)$, where $U(R \cup T)$ is a suitable regular neighbourhood of $R \cup T$. The set $\partial \mathbf{B}^3 \setminus U(R \cup T)$ is a system $\{\check{W}_1, \check{W}'_1, \dots, \check{W}_m, \check{W}'_m\}$ of $2m$ disks in ∂C . Here $\check{W}_i, \check{W}'_i$ are contained in the wings W_i, W'_i of the butterfly P_i with trunk t_i and p identifies $\check{W}_i, \check{W}'_i$. Thus M^{**} is a handlebody. Therefore $M = M^{**} \cup U(T^{**}) \cup U(R^{**}) \cup U(V^*)$. The set $U(T^{**})$ is a set of m 2-handles that are attached to the handlebody M^{**} . The attaching spheres for these 2-handles are meridians μ_1, \dots, μ_m of $p(t_1), \dots, p(t_m)$. Then μ_i cuts $p(\check{W}_i) = p(\check{W}'_i)$ transversely in just one point. Therefore $M^{**} \cup U(T^{**})$ is a 3-ball C^3 . Thus

$$M = C^3 \cup U(R^{**}) \cup U(V^*).$$

Since $U(R^{**})$ are 2-handles attached to C^3 it follows that $C^3 \cup U(R^{**})$ is a punctured 3-ball. Since the boundary of $C^3 \cup U(R^{**})$ and $U(V^*)$ coincide, it follows that $\partial U(V^*)$ is a disjoint union of spheres. From the above claim, it follows that $U(V^*)$ is a disjoint union of cones over spheres. That is, $U(V^*)$ is a disjoint union of balls. Then M is homeomorphic to S^3 .

To prove that $p(T)$ is a knot or a link, it is enough to show that $p^{-1}(p(v))$, for every A -vertex v , contains exactly two A -vertices. To prove this we construct the following graph Γ .

Assume that the 3-cell \mathbf{B}^3 is the upper half space \mathbb{R}_+^3 of $\mathbb{R}^3 + \infty$, and that the graph R lies in its boundary $\mathbb{R}^2 \times \{0\}$.

Let (P, t) be a butterfly of (R, T) and let $f_P : P_{2k} \rightarrow \bar{P}$ be its fixed parameterization. Let w_1, w_2, \dots, w_{2r} be the vertices of P_{2k} and let $v_j = f_P(w_j)$ be the vertices of ∂P . For a vertex $w_j = \cos(k\pi/r) \pm i \sin(k\pi/r)$, $k = 1, 2, \dots, r-1$, let $L(w_j)$ be the open vertical line segment $(\cos(k\pi/r) + i \sin(k\pi/r), \cos(k\pi/r) - i \sin(k\pi/r))$. For each A -vertex in ∂P not in ∂t and each E -vertex $v_j = f_P(w_j)$ in ∂P , take the arc $Q_{v_j} = f_P(L(w_j))$.

Denote by Γ the union of all possible Q_v 's for any $v \in R$ that is an A - or E -vertex.

Claim 2: Γ is a disjoint union of arcs bounded by A -vertices.

Proof: (1) Noting that if v is an A -vertex then any other vertex, related to it, is an A - or E -vertex and it follows that the vertices of Γ are all A - or E -vertices.

(2) Since by definition the A -vertices are bivalent in R and they are end points of some trunk it follows that they are monovalent vertices of Γ .

(3) Since by definition the E -vertices are bivalent in R and they are not end points of a trunk it follows that they are bivalent vertices of Γ .

Thus, each component of the graph Γ is linear and it is bounded by two A vertices.

To finish the proof of the theorem we observe that if Γ_0 is a component of Γ , the set of vertices of Γ_0 form a complete equivalence class under \simeq . Therefore, $p^{-1}(p(v))$ for every A -vertex v contains exactly two A -vertices. Hence the graph $p(T)$ is a knot or a link. \square

DEFINITION 2. *The knot or link $p(T)$ defined by the m -butterfly (R, T) will be denoted by $L(R, T)$, and we say that $L(R, T)$ has the butterfly representation (R, T) with butterfly number m , or that the m -butterfly diagram (R, T) represents $L(R, T)$.*

4. From the Butterfly to the Link

In this section we show how to construct the link $L(R, T)$ from an m -butterfly (R, T) .

Recall, [17], that a regular diagram D_L of a link L is an m -bridge diagram for the link L if we can divide up D_L into two sets of polygonal curves $O = \{o_1, o_2, \dots, o_m\}$ and $U = \{u_1, u_2, \dots, u_m\}$ ($m > 0$) such that:

- i. $D_L = o_1 \cup o_2 \cup \dots \cup o_m \cup u_1 \cup u_2 \cup \dots \cup u_m$,
- ii. o_1, o_2, \dots, o_m are mutually disjoint simple curves,
- iii. u_1, u_2, \dots, u_m are mutually disjoint simple curves,
- iv. At the crossing points of D_L , o_1, o_2, \dots, o_m are segments that pass *over* at least one crossing point, while u_1, u_2, \dots, u_m are segments that pass *under* at least one crossing point.

The arcs o_1, o_2, \dots, o_m are called *bridges* or *overarcs*. We use the notation $D_L = (O, U)$ when we want to describe explicitly the bridge presentation of the link L .

Note that, by condition iv., there are link diagrams that are not bridge diagrams. For instance, a simple closed curve is not a *bridge diagram* for the trivial knot. In this paper, we follow [2] and we differ from [18], where it is considered the trivial knot with no crossing as having an m -bridge diagram, for all $m \in \mathbb{N}$. When a link L has unknotted components, we need to take some care about them, in order to obtain an m -bridge diagram of L because no component can be expressed as a union of only o 's or u 's. Actually, we have to make at least one kink to the trivial knot to obtain a bridge diagram for it.

DEFINITION 3. *Given a link L , the bridge number of L is the minimum number m among of all possible m -bridge diagrams of the link L . It is denoted by $b(L)$.*

For example, the trivial knot has bridge number 1 (see Fig. 10c).

LEMMA 1. *Given a link L , there exists an m -bridge diagram D_L for L , such that D_L is connected and has no closed curves.*

PROOF. If the diagram has a closed circle that splits or if it is not connected, apply the moves shown in Figures 6 and 7.

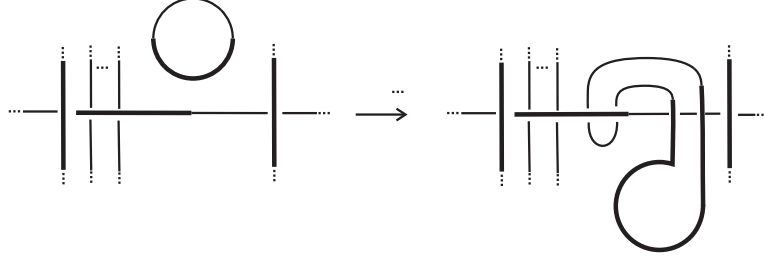


FIGURE 6. Eliminating closed curves.

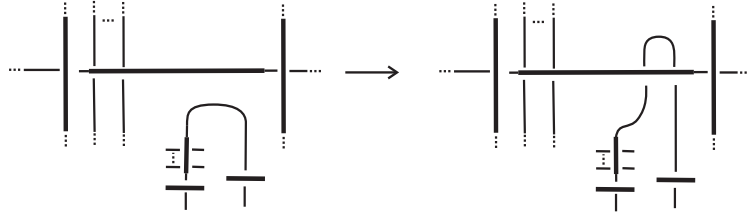


FIGURE 7. Connecting the diagram.

□

Now, given an m -butterfly diagram (R, T) we will describe an algorithm (the *butterfly-link algorithm*) to construct the link $L = L(R, T)$. Moreover, we will produce an m -bridge diagram for the link $L(R, T)$.

First of all, consider the following link K^* of \mathbb{R}_+^3 .

$$K^* = (\Gamma \times \{1/2\}) \cup (T \times \{1\}) \cup (\partial T \times [1/2, 1]),$$

where Γ is the graph defined in the proof of Theorem 1. By the second claim in the proof of Theorem 1, $\Gamma \times \{1/2\}$ is a disjoint union of arcs lying in $\mathbb{R}^2 \times \{1/2\}$. Therefore $(\Gamma \times \{1/2\}, T \times \{1\} \cup (\partial T \times [1/2, 1]))$ is an m -bridge presentation of the knot (or link) K^* . This proves the second part of Theorem 2.

In Fig. 8 we illustrate a portion of K^* . On plane $\mathbb{R}^2 \times \{0\}$ we see a component of Γ, Γ_1 , that is bounded by two components of T (denoted generically by T), whose intersection with that Γ_1 is composed of two A -vertices (denoted generically by A) and that passes through two E -vertices (denoted by E). The points f, g and h are intersections of some components of T with Γ_1 (we do not depict those components but they are transversal to Γ_1).

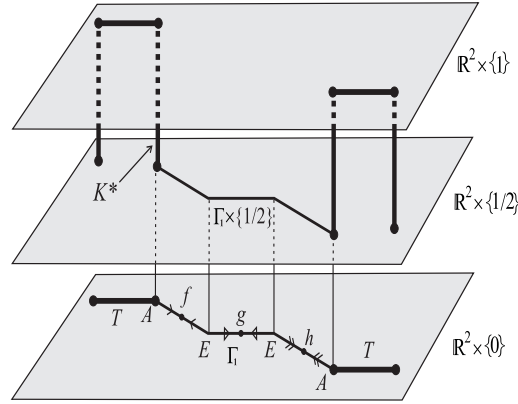


FIGURE 8. $K^* = \Gamma \times \{1/2\} \cup T \times \{1\} \cup \partial T \times [1/2, 1]$

THEOREM 2. *Given an m -butterfly diagram (R, T) the link $L(R, T)$ is isotopic to K^* . Moreover $(\Gamma \times \{1/2\}, T \times \{1\} \cup (\partial T \times [1/2, 1]))$ is an m -bridge presentation of $L(R, T)$.*

PROOF. Consider a component Γ_1 of Γ . It is linear and bounded by two A -vertices. Call $\partial\Gamma_1$ the set of these two A -vertices.

Consider the subset $\Gamma_1 \times [0, 1/2]$ of \mathbb{R}_+^3 . Then $p(\Gamma_1 \times [0, 1/2])$ is a cone $C(w, p(\Gamma_1 \times \{1/2\}))$ from the point $w = p(\Gamma_1^{(0)} \times \{0\})$ over $p(\Gamma_1 \times \{1/2\})$ (compare Figures 8 and 9) where $\Gamma_1^{(0)}$ is the set of vertices of Γ_1 . We push $p(\Gamma_1 \times \{1/2\})$ along the cone $C(w, p(\partial\Gamma_1 \times \{1/2\}))$. This we do, as shown in Fig. 9, by an isotopy H_i whose final image is just $p(\partial\Gamma_1 \times [0, 1/2])$.

Combining these isotopies H_i for all components Γ_i of Γ we obtain an isotopy H sending K^* onto the set

$$p((T \times \{1\}) \cup (\partial T \times [0, 1])).$$

But there is certainly an isotopy H' sending $p((T \times \{1\}) \cup (\partial T \times [0, 1]))$ onto $p(T \times \{0\}) = K$. This finishes the first part of the proof. \square

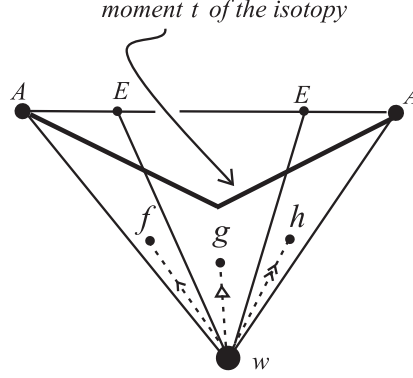


FIGURE 9. Isotopy

Algorithm (Butterfly-Link algorithm).

Finally we have:

- Start with an m -butterfly diagram on the plane $\mathbb{R}^2 \times \{0\}$. We want to construct the link $L(R, T)$.
- Construct the graph $\Gamma \subset \mathbb{R}^2 \times \{0\}$ as in the proof of the Theorem 1. See the dotted lines in Fig. 10.
- Then the link $L(R, T)$ is $(\Gamma \times \{0\}) \cup (T \times \{1\}) \cup (\partial T \times [0, 1])$.
- And $(\Gamma \times \{0\}, T \times \{1\} \cup (\partial T \times [0, 1]))$ is an m -bridge diagram of $L(R, T)$.

EXAMPLE 2. Applying the butterfly-link algorithm found in the proof of Theorem 2 to the three butterfly diagrams of Fig. 5 we obtain the knots of Fig. 10. The knot of Fig. 10a is the knot 4_1 , the knot of Fig. 10b is the knot 8_{20} and the knot in 10c is the trivial knot.

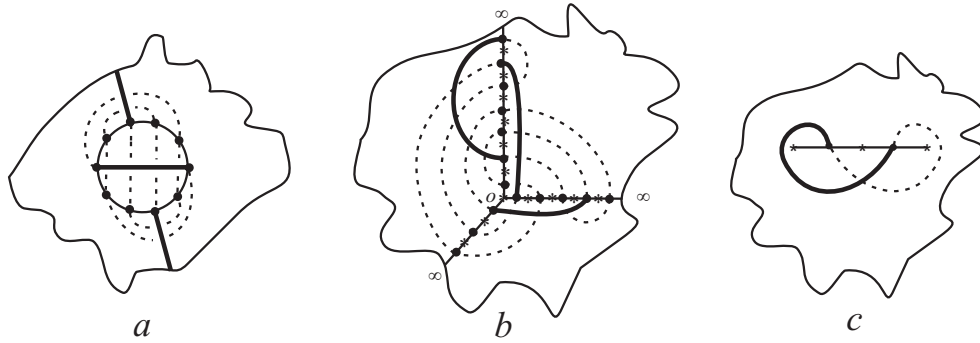


FIGURE 10. Examples of knots produced by the butterfly-link algorithm.

5. From Links to Butterflies

Now, in the other direction, we explain how to obtain a butterfly from a given link.

THEOREM 3. *Every knot or link can be represented by an m -butterfly diagram, for some $m > 0$. Moreover the m -butterfly can be chosen with no E -vertices.*

PROOF. Given a link L , let D_L be an m -bridge diagram of L , connected. See Fig. 11. Usually, in the theory of knots, we do not draw the dotted lines. We assume that they are under the plane $\mathbb{R}^2 \times \{0\}$ and so the diagram can be seen as a finite collection $T = \{t_1, \dots, t_m\}$ of disjoint arcs (no closed curves) in the plane $\mathbb{R}^2 \times \{0\}$. Select a point B_i in each one of the *regions* of the complement of D_L in $\mathbb{R}^2 \times \{0\}$. For the unbounded component, set $B_0 = \infty$.

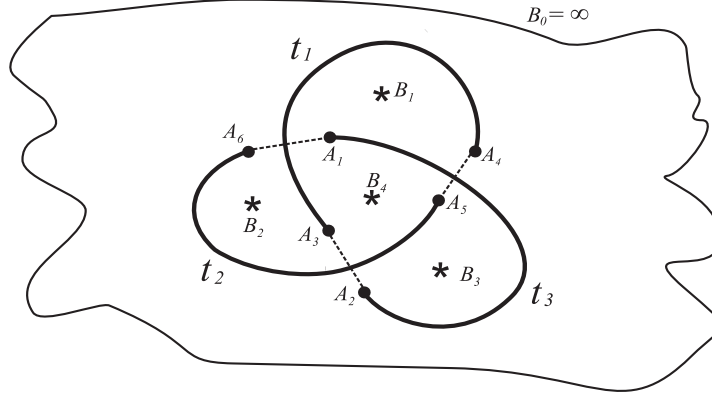


FIGURE 11. Regions of $\mathbb{R}^2 \setminus D_L$.

The boundary points of the arcs t_i of the link-diagram D_L will be the A -vertices of our m -butterfly diagram.

Each A -vertex belongs to the boundary of two regions. The vertices denoted by B (and selected before) in these two regions will be called the *neighboring B 's of the A -vertex*. (In Fig. 12, the neighboring B 's of the A -vertex A_1 are B_1 and B_4 .)

The diagram D_L contains also *crossings*. A crossing involves an overarc and two adjacent *arcs*.

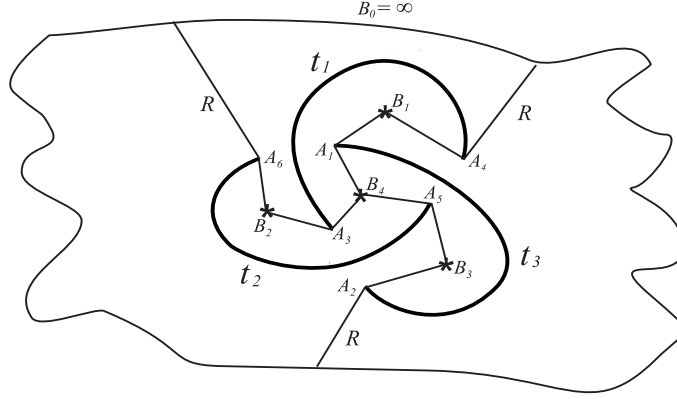
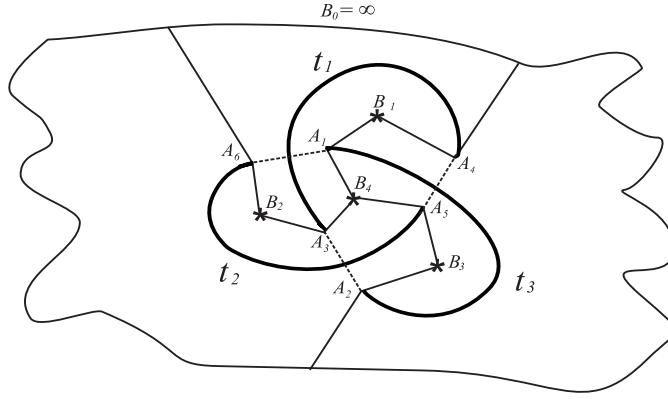
We now proceed to construct an m -butterfly diagram (R, T) . Joint every A -vertex of D_L with its two neighboring B 's by arcs lying in the regions to which these two belong. Thus we obtain a set of arcs R and we assume that these arcs have mutually disjoint interiors among themselves and with the arcs of T .

Then (R, T) is an m -butterfly diagram, where m is the number of arcs in T . The graph R is connected because the diagram D_L is connected. Moreover, $S^2 \setminus R$ is a disjoint union of open 2-cells, namely, open neighbourhoods of the arcs t_i of the diagram. Finally the A -vertices are bivalent in R . Note that there are no E -vertices in R . The set of B -vertices of the m -butterfly diagram is the set of B 's.

Applying the butterfly-link algorithm found in the proof of Theorem 2 to (R, T) (here the graph Γ is the set of dotted lines), it is easy to see that $L = L(R, T)$, see Fig. 13.

□

We will refer to the algorithm described in the proof of Theorem 3 as the *link-butterfly algorithm*.

FIGURE 12. m -butterfly from a link-diagram D_L .FIGURE 13. Link $L(R, T)$ from an m -butterfly diagram.

DEFINITION 4. *The minimum m among all possible m -butterfly diagrams of a given link L is called the butterfly number of L and it is denoted by $m(L)$.*

For example, the butterfly number of the trivial knot is 1, see Fig. 5 c; the butterfly number of any rational knot is 2, see the Introduction and Fig. 22; and the butterfly number of the borromean rings is 3, see Fig. 27.

6. Trunk-reducing Move

Our goal in the next two sections is to prove that the butterfly and bridge number of knots and links coincide. To achieve this we need to know how to reduce the number of trunks obtained by the link-butterfly algorithm described in Section 5.

Let L be a link and (R, T) be an m -butterfly diagram of L found by the link-butterfly algorithm. We observed that it does not produce E -vertices. Actually it produces only two types of butterflies. The butterflies, coming from trunks that are overarcs, have more than two A -vertices, as illustrated in Fig. 14a. The butterflies

coming from trunks that are not overarcs (simple arcs) have only two A -vertices. We call this last kind of butterflies *simple butterflies*. They have the shape illustrated in Fig. 14b.

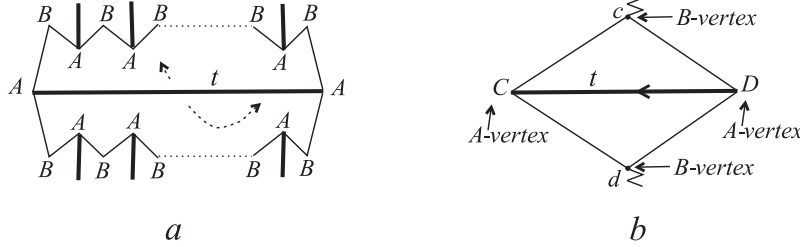


FIGURE 14. a. A non simple butterfly.

b. A simple butterfly.

We also notice that the value of m in the m -butterfly diagram (R, T) is just the number of all arcs in the chosen link diagram.

So given a connected m -bridge diagram of a link L , together with the m -butterfly diagram (R, T) representation of L produced using the link-butterfly algorithm, a natural question arises:

Is it possible to make some *moves* on the m -butterfly diagram (R, T) , in such a way, that we find a different l -butterfly diagram (R', T') of L but with $l < m$? We will see that we can do this, but at the expense of producing E -vertices.

Now we will show how to decrease the number of butterflies in a given m -butterfly. More specifically, trunks of simple butterflies will be converted into E -vertices.

Let P be the simple butterfly of (R, T) shown in Fig. 15, where the vertex labeled by D at the rightmost part of the Figure is an A - or E -vertex and the vertex labeled by C at the leftmost part of the Figure is an A -vertex.

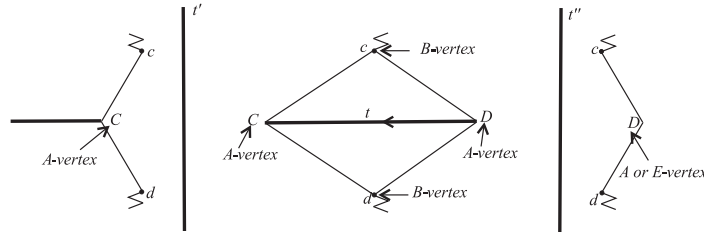


FIGURE 15. Simple butterfly

For simplicity, we will assume here that the closed 3-cell of (R, T) is below the paper. Consider the notations given in Fig. 15. On both sides of the trunk t' we draw the arcs $C'c$ and $C'd$ (See Fig. 16). We use the same notation on both sides, to indicate that they match by the "reflection" along t' . Inside the 3-cell we trace an arc $C'D$ getting two triangles $C'cD$ and $C'dD$ that have only two edges on the boundary of (R, T) . These triangles together with the wings CcD and CDd of the simple butterfly on the boundary of $\partial \mathbf{B}$ can be considered as the boundary of a pyramid with quadrilateral base $CcC'd$ and apex D .

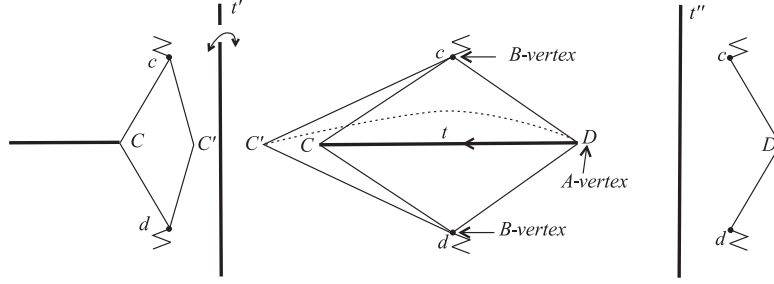
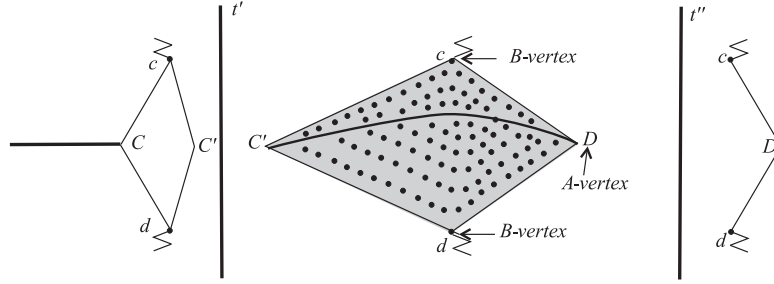
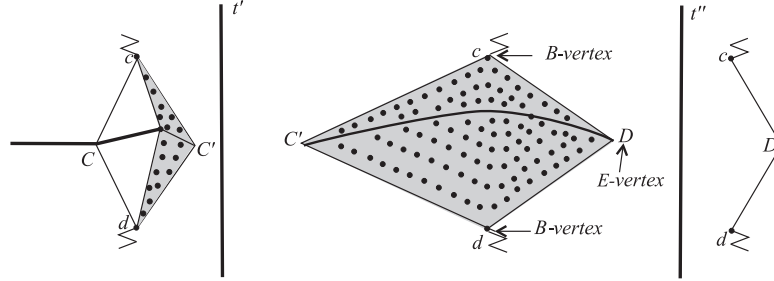


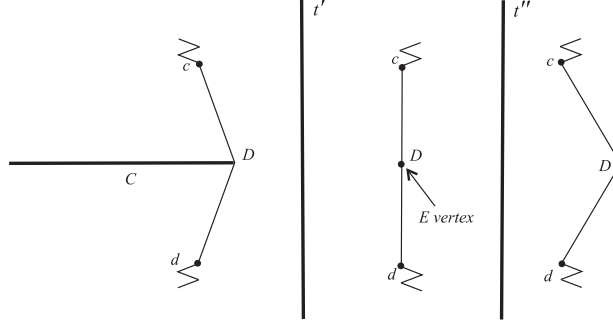
FIGURE 16. First step

FIGURE 17. Cutting off $CcC'dD$

Now we cut the pyramid $CcC'dD$ out of the ball (R, T) (Fig. 17) and glue it on the other side of t' to the corresponding base $CcC'd$, thus obtaining finally Fig. 18.

FIGURE 18. Gluing $CcC'dD$

In this way the simple butterfly has been substituted by two edges cD and dD and a E -vertex D (see Fig. 19). In this process the graph R becomes a connected graph R_1 such that $S^2 \setminus R_1 = S^2 \setminus (R \cup \bar{P})$, where P is the simple butterfly of (R, T) shown in Fig. 15. Hence $S^2 \setminus R_1$ consists of a disjoint union of open 2-cells. Therefore R_1 together with the new collection of trunks T_1 is in fact a butterfly diagram. Moreover, note that the new E -vertex D is bivalent in R . See the center part of Fig. 19. The point C is not any more an A -vertex (actually, it is now a

FIGURE 19. A new E -vertex

point in the interior of a trunk, (See the leftmost part of Fig. 19), and notice that the valence of the B -vertices of the simple butterfly P decreases by one. Recall that a vertex of R is a B -vertex iff $p^{-1}(p(v))$ contains at least one non-bivalent vertex, where $p : \mathbf{B}^3 \rightarrow M(R, T)$ is the identification map. So, it is possible that some of the B -vertices are not any more B -vertices but it is not a problem since they can be considered as any other point in R_1 that is not a vertex.

The transition from Fig. 15 to Fig. 19 will be referred to as a “*trunk-reducing move*”.

We have proved the following theorem

THEOREM 4. *A trunk-reducing move converts an m -butterfly diagram of a link L into an $(m - 1)$ -butterfly diagram of the same link L . The new diagram gets a new E -vertex in place of a simple butterfly.*

PROOF. It is enough to apply the butterfly-link algorithm to both butterfly diagrams. Apply it to Figures 15 and 19. \square

EXAMPLE 3. *Let us apply trunk-reducing moves to the 4-butterfly diagram of the trefoil knot illustrated in Fig. 20. There, we have four trunks: t_1, t_2, t_3, t_4 , and six B -vertices; a, b, c, d, e, f corresponding to each region of the diagram of the knot. For simplicity, we do not draw the edges joining A - and B -vertices of the corresponding butterfly (R, T) .*

The arcs t_1 and t_4 correspond to simple butterflies. Therefore, performing two trunk-reducing moves in t_1 and t_4 (in this order), the trunks t_1 and t_4 are reduced to the E -vertices labeled by E_1 and E_4 , respectively (see Fig. 21). The diagram of Fig. 21 is not yet a butterfly diagram because it contains too many vertices. Indeed, under the application of the trunk-reducing moves the B -vertices of the original diagram become bivalent vertices of the new diagram that are not A -vertices nor E -vertices. Therefore we can delete them, thus obtaining the 2-butterfly diagram of Fig. 22.

A 4-butterfly diagram for the trivial link with two components is depicted in Fig. 23.

Applying one trunk-reducing move we get Fig. 24. A second trunk-reducing move produces the 2-butterfly diagram representing the trivial link with two components shown in Fig. 25a. In Fig. 25b we apply the butterfly-link algorithm to the 2-butterfly to recover the link.

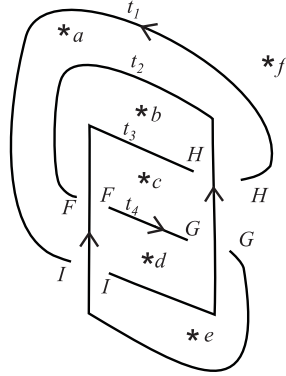


FIGURE 20. A 4-butterfly representation of the trefoil knot.

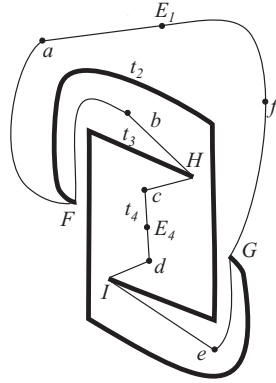
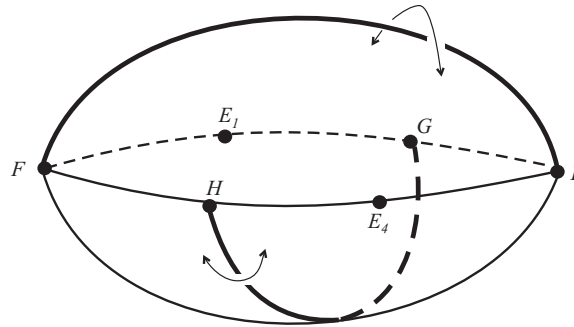
FIGURE 21. Diagram with new E -vertices.

FIGURE 22. A 2-butterfly representation of the trefoil knot.

REMARK 1. *The inverse of a trunk-reducing move can certainly be applied to any E -vertex in an m -butterfly diagram to increase the number of trunks. In this*

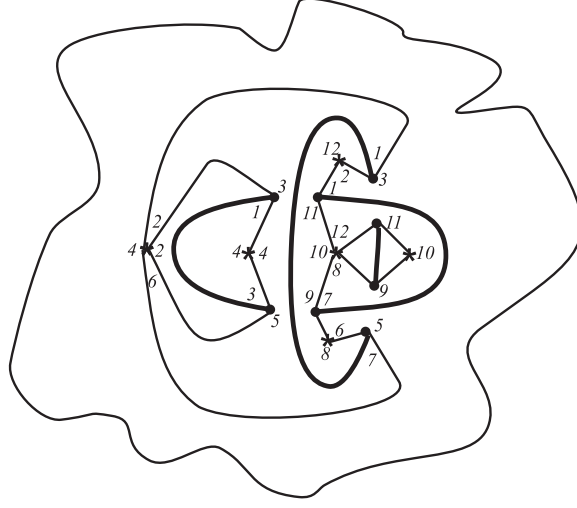


FIGURE 23. A 4-butterfly for the trivial link with two components.

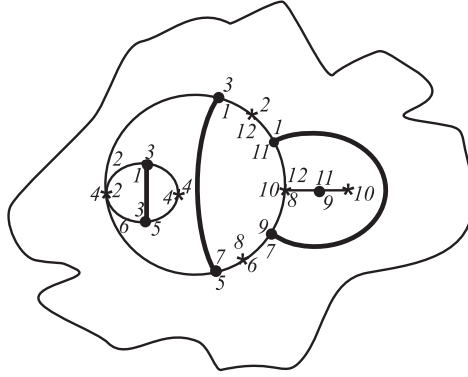


FIGURE 24. A 3-butterfly for the trivial link with two components, obtained by a trunk-reducing move.

way it is always possible to obtain a butterfly diagram without E -vertices from any given butterfly diagram of a link.

7. The Bridge Number and the Butterfly Number

Let us remark that the knot-diagram of the trefoil knot given in Example 3 corresponds to a 2-bridge presentation of it and by applying trunk-reducing moves we obtained a 2-butterfly diagram of the trefoil knot. Actually this is a general result, and we want to show that for any link L , the butterfly number equals the bridge number, i.e., $m(L) = b(L)$.

THEOREM 5. *For any link L , $b(L) = m(L)$.*

PROOF. The fact that $b(L) \leq m(L)$ is a corollary of Theorem 2. Now we will show that $m(L) \leq b(L)$ for any link L .

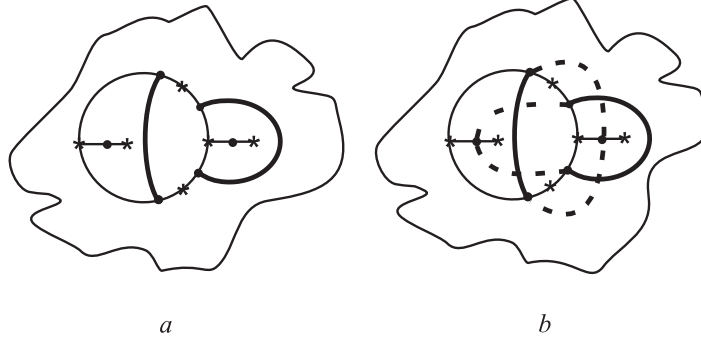


FIGURE 25. A 2-butterfly diagram representing the trivial link with two components.

Let D_L be a link-diagram of L , such that it satisfies the conditions of Lemma 1 and the number of bridges (or overarcs) is $b(L)$.

We can apply the link-butterfly algorithm to D_L to obtain an m -butterfly diagram (R, T) without E -vertices, where m is the number of arcs of D_L (Theorem 3).

Next apply trunk-reducing moves to (R, T) in order to trade simple butterflies by pairs of edges and E -vertices. We have to be careful because we cannot apply the trunk-reducing moves at random. (Remember that to be able to apply a trunk-reducing move we need that one of the two neighbouring vertices be an A -vertex.) To have a consistent order of application for a component L_i of L , we start with an overarc of the projection of L_i (granted by Proposition 1) and we tour L_i , following some orientation, performing trunk-reducing moves to the simple butterflies in the same order that they are found. In this way we eliminate all the simple arcs belonging to L_i and convert them into E -points. We do this for every component of L . Therefore all simple butterflies disappear (converted into E -vertices) and there remains only the trunks coming from overarcs. Since the number of overarcs of D_L is $b(L)$ the new butterfly is a $b(L)$ -butterfly diagram. Then $m(L) \leq b(L)$. \square

EXAMPLE 4. Consider the 3-bridge presentation of the borromean rings given in Fig. 26. Make trunk-reducing moves first to the sequence t_2, t_3, t_4 . Next to the sequence t_6, t_7, t_8 , and finally to the sequence t_{10}, t_{11}, t_{12} . You will get the 3-butterfly diagram of Fig. 27, where those trunks have been exchanged by the E -vertices A, B, C, D, E, F, G, H , and I , respectively. The vertices $o, \infty, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, and 12 are B -vertices and all of them belong to the orbit of $\{o\}$, under the equivalence relation \simeq .

Another way to visualize this 3-butterfly diagram is shown in Fig. 28, where, for simplicity, we do not mark the B -vertices, except o and ∞ .

8. Conclusions

We have proved that any link can be represented as an m -butterfly. We defined the butterfly number of a link and we proved that the butterfly number equals the bridge number of a link. Therefore it is feasible to study the m -bridge links via m -butterflies. For each 2-bridge link the associated 2-butterfly allow us to visualize

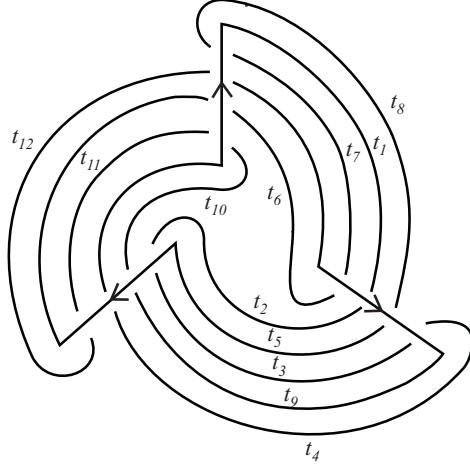


FIGURE 26. A 3-bridge presentation.

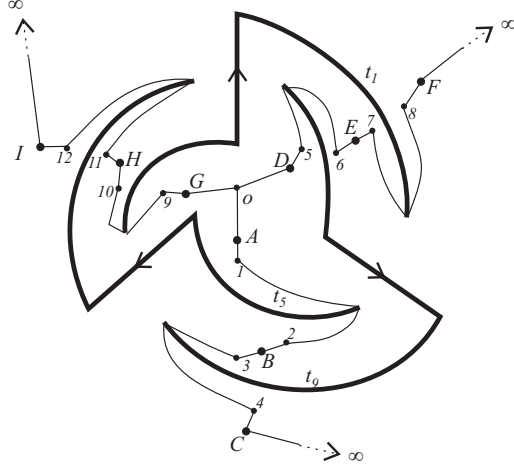


FIGURE 27. A 3-butterfly.

the corresponding rational number. For example, in Fig. 5a we have a 2-butterfly that represents the rational knot $5/2$. For the 3-bridge links, as we have announced in the introduction, it is possible to associate a set of 3 rational numbers to each 3-butterfly. For more details about the way to assign a set of three rational numbers to a 3-butterfly diagram see [22], [11].

For example, in Fig. 29 we show the diagrams of two 3-butterflies, (R_1, T_1) and (R_2, T_2) , with the associated set of rational numbers.

The two diagrams are different, however $L(R_1, T_1)$ and $L(R_2, T_2)$ are equivalent 3-bridge presentation of the knot 8_{20} with bridge (and butterfly) number 3. To exhibit the equivalence between the bridge presentations $L(R_1, T_1)$ and $L(R_2, T_2)$ we modify the presentation of the two 3-butterfly diagrams (R_1, T_1) and (R_2, T_2) ,

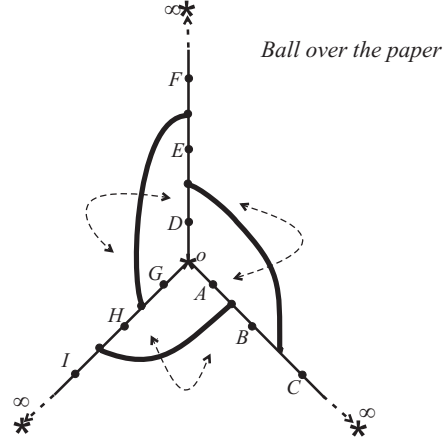


FIGURE 28. A 3-butterfly diagram for de borromean rings (without some B-vertices).

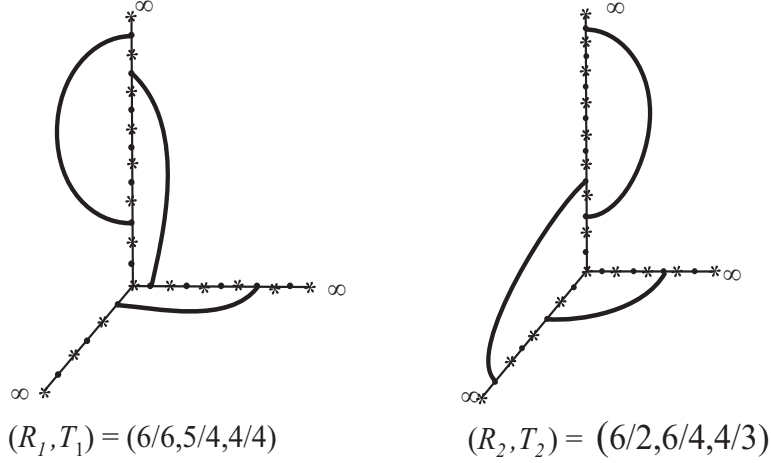


FIGURE 29. Two butterfly diagrams for the knot 8_{20} , with the associate rational numbers.

as shown in Fig. 30, on the left. In the center we have the link diagrams obtained when we close the 3-butterflies.

Then we move the dotted arc as shown in each diagram.

This raises the problem of *finding a set of moves* in a butterfly diagram connecting diagrams representing the same link. This is left as an open problem.

References

- [1] G. Burde and H. Zieschang. *Knots*. Walter de Gruyter, New York, NY (1985).
- [2] J. W. Chung, X. S. Lin, On the bridge number of knot diagrams with minimal crossings. *Math.Proc.Camb.Philos.Soc.* **137**, No. 3 (2004) 617-632.
- [3] P. Cromwell, *Knots and Links*, Cambridge University Press, Cambridge (2004).

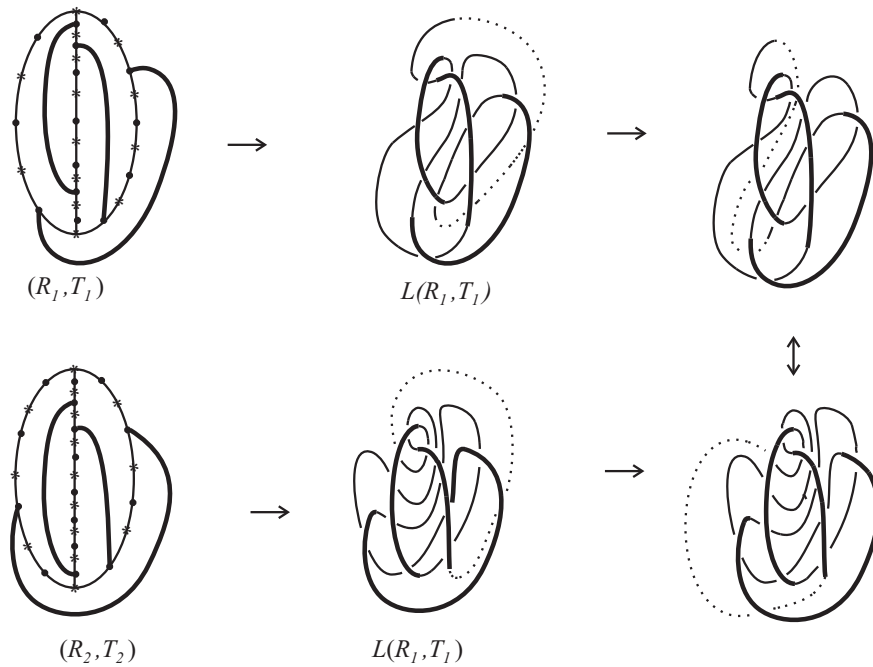


FIGURE 30. Two non equivalent 3-butterfly diagram with equivalent 3-bridge presentation.

- [4] I. Fáry, On the straight line representations of planar graphs, *Acta Sci. Math.* **11** (1948) 229-233.
- [5] R. H. Crowell; R. H. Fox. *Introduction to knot theory*. Reprint of the 1963 original. Graduate Texts in Mathematics, **57**. Springer-Verlag, New York-Heidelberg (1977).
- [6] A. Edmonds; C. Livingston. Symmetric representations of knot groups. *Topology Appl.* **18**, No. 2-3 (1984) 281-312.
- [7] H. M. Hilden. Every closed orientable 3-manifold is a 3-fold branched covering space of S^3 . *Bull. Amer. Math. Soc.* **80** (1974) 1243-1244.
- [8] H. M. Hilden, J. M. Montesinos, D. M. Tejada and M. M. Toro. Fox coloured knots and triangulations of S^3 . *Math. Proc. Camb. Phil. Soc.* **141**, No. 3 (2006) 443-463.
- [9] H. M. Hilden, J. M. Montesinos, D. M. Tejada and M. M. Toro. Representing 3-manifolds by triangulations of S^3 . *Revista Colombiana de Matemáticas*, **39**, No 2 (2005) 63-86.
- [10] H. M. Hilden, J. M. Montesinos, D. M. Tejada and M. M. Toro. Mariposas y 3-variedades. *Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales*, **28** (2004) 71-78.
- [11] H. M. Hilden, J. M. Montesinos, D. M. Tejada and M. M. Toro. On a classification of 3-bridge links. Preprint, 2010.
- [12] H. M. Hilden, J. M. Montesinos, D. M. Tejada and M. M. Toro. Virtual knots and butterflies. Preprint, 2010.
- [13] Kauffman, L. Virtual Knot theory, *Europ. J. Combinatorics* **20** (1999) 663-691.
- [14] R. Lickorish. *An Introduction to Knot Theory*. Graduate texts in Mathematics **175**, Springer-Verlag, New York (1997).
- [15] J. M. Montesinos. A representation of closed orientable 3-manifolds as 3-fold branched coverings of S^3 . *Bull. Amer. Math. Soc.* **80** (1974) 845-846.
- [16] J. M. Montesinos. Lectures on 3-fold simple coverings and 3-manifolds. *Combinatorial methods in topology and algebraic geometry* (Rochester, N.Y., 1982), 157-177, Contemp. Math., 44, Amer. Math. Soc., Providence, RI, 1985.
- [17] K. Murasugi, *Knot theory and its applications*. Birkhäuser, Boston (1996).

- [18] S. Negami and K. Okita, The splittability and triviality of 3-bridge links, *Trans. Amer. Math. Soc.* **289**, No 1 (1985) 253-280.
- [19] H. Seifert and W. Threlfall, *A textbook of topology*, Pure and Applied Mathematics, **89**, Academic Press Inc., Harcourt Brace Jovanovich, New York-London (1980).
- [20] H. Schubert, Knoten mit zwei Brücken, *Math. Z.* **65** (1956) 133-170.
- [21] D. M. Tejada, Variedades Triangulaciones y Representaciones, Universidad Nacional de Colombia, Sede Medellín, Medellín (2004).
- [22] M. M. Toro, Nudos Combinatorios y Mariposas, *Rev. Acad. Colomb. Cienc.* **28**, **106** (2004) 79-86.
- [23] M.M. Toro, Enlaces de tres puentes y mariposas, Universidad Nacional de Colombia, Sede Medellín, Medellín (2010).
- [24] W. Thurston, *Geometry and Topology of 3-Manifolds*, xeroxed notes from Princeton University (1978).
- [25] W. Thurston, Three dimensional manifolds, Kleinian groups, and hyperbolic geometry, *Bull. Amer. Math. Soc., New Ser.* **6**, (1982) 357-379.

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